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Matching-star Ramsey sets

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Abstract

The Ramsey set $\mathscr{R}(G,H)$ consists of all graphs F with $F \to (G,H)$ and $F' \nrightarrow (G,H)$ for every proper subgraph F' of F. In this paper we will characterize the graphs belonging to $\mathscr{R}(2K_2, K_{1,n})$ with $n \geqslant 3$ and determine $\mathscr{R}(2K_2, K_{1,n})$ for $n \leqslant 3$ explicitly. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

If F, G and H are (simple) graphs, $F \to (G,H)$ means that in any 2-coloring of the edges of F with red and green there is a red subgraph isomorphic to G or a green subgraph isomorphic to H. For a given pair of graphs (G,H) the Ramsey set $\mathscr{R}(G,H)$ is defined to be the set of all graphs F (up to isomorphism) with $F \to (G,H)$ and $F' \nrightarrow (G,H)$ for every proper subgraph F' of F.

In [2] it has been shown that $\mathscr{R}(mK_2, H)$ is finite for arbitrary H, but a complete determination seems to be very difficult except for some small-order graphs H and some small m. Some results have been obtained for the first non-trivial case $m=2$, for example $\mathscr{R}(2K_2, nK_2)$ has been studied in [1,4]. Here we will focus on $\mathscr{R}(2K_2, K_{1,n})$. We will determine these sets for $n \leqslant 3$ and present a characterization of the graphs belonging to $\mathscr{R}(2K_2, K_{1,n})$ for $n \geqslant 3$. The sets $\mathscr{R}(2K_2, K_{1,n})$ for $n \leqslant 3$ can be found also in [1] where they are given without proof.

All notation not specifically mentioned will follow that in [3].

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2. Properties of graphs in $\mathscr{R}(2K_2, K_{1,n})$

As usual, V and E are used to denote the vertex-set and the edge-set of a graph G. The degree of a vertex v is denoted by $d(v)$ and the maximum degree in G by $\Delta(G)$. Moreover, a $(2K_2, K_{1,n})$-coloring is a 2-coloring where neither a red $2K_2$ nor a green $K_{1,n}$ occurs.

Lemma 1. *Let $F \in \mathscr{R}(2K_2, K_{1,n})$. Then F is isolate-free and every edge is incident to at least one vertex of degree at least n.*

Proof. The minimality of F implies that F is isolate-free. Suppose now that $d(u)$, $d(v) < n$ for some $e = uv \in E(F)$. Then take a $(2K_2, K_{1,n})$-coloring of $F - e$ (which must exist because of the minimality of F) and color the edge e green. This yields a $(2K_2, K_{1,n})$-coloring of F, a contradiction. $\square$

Lemma 2. *$F \rightarrow (2K_2, K_{1,n})$ holds iff the following conditions are satisfied*:
(i) *$K_{1,n} \subseteq F - v$ for every $v \in V(F)$ and*
(ii) *$K_{1,n} \subseteq F - E(C_3)$ for every cycle C_3 in F.*

Proof. First suppose that (i) is violated for some $v \in V(F)$ or (ii) for some C_3 in F. In the first case color all edges incident to v red and in the second one the edges of the C_3. In both cases a $(2K_2, K_{1,n})$-coloring of F is obtained if the remaining edges are colored green.

Suppose now that (i) and (ii) are satisfied. Consider any 2-coloring of F not containing a red $2K_2$. Then either no red edge occurs or the red edges form a star or a cycle C_3. In all three cases the existence of a green $K_{1,n}$ is implied by (i) or (ii). $\square$

Lemma 3. *Let $F \in \mathscr{R}(2K_2, K_{1,n})$, $e \in E(F)$, and $U_e = \{v \in V(F) : d_{F-e}(v) \geqslant n\}$. Then there must be a star in $F - e$ containing all vertices belonging to U_e.*

Proof. This follows from Lemma 2 because of $F - e \nrightarrow (2K_2, K_{1,n})$. $\square$

Lemma 4. *Let $F \in \mathscr{R}(2K_2, K_{1,n})$. Then $n \leqslant \Delta(F) \leqslant n + 1$.*

Proof. Lemma 2(i) implies $\Delta(F) \geqslant n$. Suppose now that $\Delta(F) \geqslant n+2$ and let $v \in V(F)$ with $d(v) = \Delta$. By Lemma 2(i), a vertex u incident to at least n edges not incident to v must exist. But this yields a contradiction to the minimality of F. Again using Lemma 2 we obtain that $F - uv \rightarrow (2K_2, K_{1,n})$ if $uv \in E(F)$ and $F - e \rightarrow (2K_2, K_{1,n})$ for every edge e incident to v if $uv \notin E$. $\square$

Lemma 5. *Let $F \in \mathscr{R}(2K_2, K_{1,n})$ with $\Delta(F) = n \geqslant 3$, and let α be the number of vertices of degree n in F. Then*

(i) $\alpha \geqslant 2$, *where equality holds if and only if* $F = 2K_{1,n}$, *and*
(ii) $\alpha \leqslant n+2$.

Proof. (i) Lemma 2(i) implies $\alpha \geqslant 2$, and it is easy to see that $2K_{1,n}$ is a member of $\mathscr{R}(2K_2, K_{1,n})$ with $\Delta = n$ and $\alpha = 2$. Suppose now that a further graph $F \in \mathscr{R}(2K_2, K_{1,n})$ with these properties exists. Let u and v be the two vertices of degree n. It follows from Lemma 2(i) that u and v are not adjacent and that a common neighbor of u and v in F is impossible. This yields a proper subgraph $2K_{1,n}$ in F, a contradiction to the minimality of F.

(ii) Let $u_1, \ldots, u_\alpha$ be the vertices of degree n in F. Suppose first that a vertex $v \in V$ with $d(v) \leqslant n-1$ exists. By Lemma 1 there must be an edge $vu_i \in E$, say vu_1. Then Lemma 3 implies the existence of a star in F containing the vertices $u_2, \ldots, u_\alpha$. This guarantees a vertex w in F with $d(w) \geqslant \alpha - 2$, and $\alpha \leqslant d(w) + 2 \leqslant n+2$ follows. The remaining case is that F is regular of degree n, i.e., $V = \{u_1, \ldots, u_\alpha\}$. We may assume that $u_1 u_2 \in E$. By Lemma 3 there must be a star in F containing the vertices $u_3, \ldots, u_\alpha$. This yields a vertex u with $d(u) \geqslant \alpha - 3$ in F implying $\alpha \leqslant n+3$. Suppose that $\alpha = n+3$. Then the complement $\overline{F}$ of F contains only vertices of degree 2 and consists of disjoint cycles $C^1, \ldots, C^s$. In case of $s \geqslant 2$ consider vertices $v \in C^1$ and $w \in C^2$. Then $vw \in E$ and in contradiction to Lemma 3 there is no star in F containing all vertices different from v and w. It remains that $s = 1$, i.e., $\overline{F} = C^1$. In this case C^1 has length at least 6 since $n \geqslant 3$ implies $\alpha \geqslant 6$. Thus, we can find vertices v and w with distance at least 3 in $\overline{F}$. Again in contradiction to Lemma 3 there is no star in F containing all vertices different from v and w. $\square$

3. The set $\mathscr{R}(2K_2, K_{1,n})$

Trivially, $\mathscr{R}(2K_2, K_{1,n}) = \{2K_2\}$ for $n = 1$, and for $n = 2$ we obtain:

Theorem 1. $\mathscr{R}(2K_2, K_{1,2}) = \{2K_{1,2}, C_4, C_5\}$.

Proof. It is easy to see that $2K_{1,2}$, C_4 and C_5 belong to $\mathscr{R}(2K_2, K_{1,2})$. Suppose now that there is a further graph F belonging to this Ramsey set. By the minimality of F, there are no isolates and no subgraph isomorphic to $2K_{1,2}$, C_4 or C_5 in F. Thus, F has to be connected, since every component must contain a $K_{1,2}$ by Lemma 1. Moreover, a cycle of length greater than 3 is impossible in F.

Suppose first that F is cycle-free, i.e. F is a tree. Then there must be a vertex u with $d(u) \geqslant 3$ since otherwise F would be a path yielding that $F \nrightarrow (2K_2, K_{1,2})$ or $2K_{1,2} \subset F$. One of the (at least three) components of $F - u$ has to contain a $K_{1,2}$ by Lemma 2. But this yields a $2K_{1,2}$ in F together with u und two suitable neighbors of u.

The remaining case is that F contains a cycle C of length 3. Let u_1, u_2 and u_3 be the vertices of C. Since no cycle of length greater than 3 occurs in F, the graph $F - E(C)$ consists of three components S_1, S_2 and S_3 each containing one of the vertices of C.

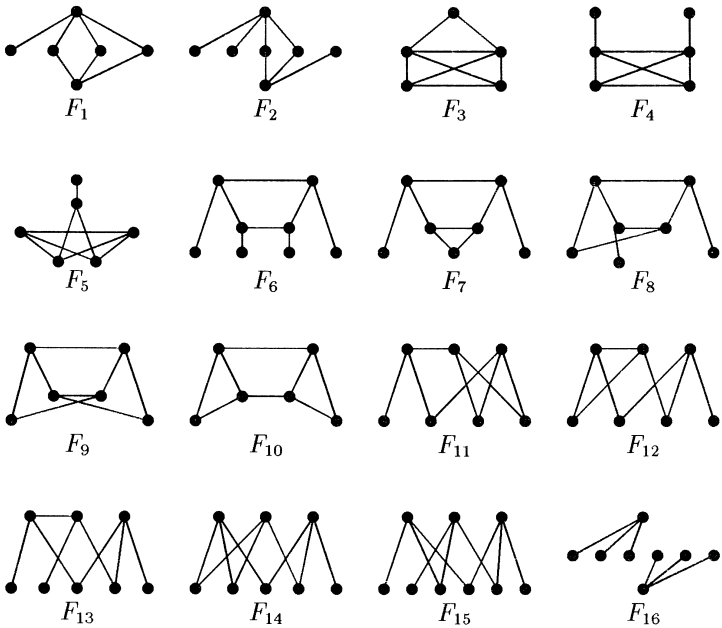

Fig. 1. The members of $\mathscr{R}(2K_2, K_{1,3})$.

We may assume that u_i belongs to S_i. By Lemma 2, one of the three components, say S_1, contains a $K_{1,2}$. A $K_{1,2}$ in S_1 not containing u_1 yields a $2K_{1,2}$ in F together with two edges of C. But if every $K_{1,2}$ in S_1 contains u_1, then there must be a $K_{1,2}$ in $F - u_1$ containing no vertex from S_1 by Lemma 2. Again we obtain a $2K_{1,2}$ in F, a contradiction. □

Now some further notation will be useful. By $\mathscr{A}_n$ we denote the class of graphs obtained from two disjoint stars $S_1 = K_{1,n+1}$ and $S_2 = K_{1,n}$ by identifying $l \geqslant 2$ vertices of degree 1 from S_1 with l vertices of degree 1 from S_2 (for example $\mathscr{A}_3 = \{F_1, F_2\}$ with F_1, F_2 from Fig. 1). For $n \geqslant 3$, the class of isolate-free graphs F of size $4n-4$ containing a subgraph K_4 with two vertices of degree $n+1$ and two vertices of degree n in F is denoted by $\mathscr{B}_n$ (for example $\mathscr{B}_3 = \{F_3, F_4\}$ with F_3, F_4 from Fig. 1). For $n \geqslant 5$, the class of isolate-free graphs F of size $4n-3$ containing an induced subgraph $K_5 - e$ with two adjacent vertices of degree $n+1$ and the two non-adjacent vertices of degree n in F is denoted by $\mathscr{C}_n$. For $n \leqslant 4$ we put $\mathscr{C}_n := \emptyset$. Moreover, we will use $\mathscr{R}'(2K_2, K_{1,n}) := \{F \in \mathscr{R}(2K_2, K_{1,n})\colon \Delta(F) = n+1\}$ and $\mathscr{R}''(2K_2, K_{1,n}) := \{F \in \mathscr{R}(2K_2, K_{1,n})\colon \Delta(F) = n\}$. Note that Lemma 4 implies

$$\mathscr{R}(2K_2, K_{1,n}) = \mathscr{R}'(2K_2, K_{1,n}) \cup \mathscr{R}''(2K_2, K_{1,n}).$$

Theorem 2. *If $n \geqslant 3$ then $\mathscr{R}'(2K_2, K_{1,n}) = \mathscr{A}_n \cup \mathscr{B}_n \cup \mathscr{C}_n$.*

Proof. First, we will prove that any graph $F \in \mathscr{A}_n \cup \mathscr{B}_n \cup \mathscr{C}_n$ must belong to $\mathscr{R}'(2K_2, K_{1,n})$. Using Lemma 2 we see that $F \to (2K_2, K_{1,n})$. The minimality of F, i.e. $F - e \nrightarrow (2K_2, K_{1,n})$ for every $e \in E(F)$, can be obtained as follows.

Assume first that $F \in \mathscr{A}_n$ and let $u, w \in V(F)$ such that $d(u) = n + 1$ and $d(w) = n$. Note that every edge in F is incident to u or to w. If e is incident to u then $K_{1,n} \nsubseteq (F - e) - v$ for a common neighbor v of u and w in $F - e$ (which must exist). If e is incident to w then $K_{1,n} \nsubseteq (F - e) - u$. Using Lemma 2 we obtain the minimality of F.

Now let $F \in \mathscr{B}_n \cup \mathscr{C}_n$. First consider the case $n \neq 4$. Then there are exactly two vertices u_1 and u_2 of degree $n + 1$ and two vertices w_1 and w_2 of degree n in F. Moreover, every edge in F is incident to one of these four vertices. If e is incident to w_1 or to w_2, say to w_1, then $K_{1,n} \nsubseteq (F - e) - E(C_3)$ for the cycle C_3 containing the vertices u_1, u_2 and w_2. If e is incident to u_1 or to u_2, say to u_1 and $e \neq u_1u_2$ then $K_{1,n} \nsubseteq (F - e) - u_2$. If $e = u_1u_2$ and $F \in \mathscr{B}_n$ then $K_{1,n} \nsubseteq (F - e) - w_1$. If $e = u_1u_2$ and $F \in \mathscr{C}_n$ then $K_{1,n} \nsubseteq (F - e) - v$ where v is a common neighbor of u_1, u_2, w_1 and w_2 (at least one such vertex exists). Again Lemma 2 yields the minimality of F. The remaining case is $n = 4$ and $F \in \mathscr{B}_n$. If then there are only four vertices of degree at least 4 in F we can proceed as in the case $n \neq 4$. Otherwise F has to consist of a K_5 and two further edges which are incident to two vertices of the K_5. Again the minimality can be checked easily using Lemma 2.

It remains to show that any graph $F \in \mathscr{R}'(2K_2, K_{1,n})$ belongs to $\mathscr{A}_n \cup \mathscr{B}_n \cup \mathscr{C}_n$. Let u_1 be a vertex in F with $d(u_1) = n + 1$. One of the following two cases must occur.

Case I: F contains a vertex w non-adjacent to u_1 with $d(w) \geqslant n$. Then u_1 and w must have at least two common neighbors since otherwise $2K_{1,n}$ would be a proper subgraph of F in contradiction to the minimality of F (see Lemma 5). Thus, F contains a subgraph F' belonging to $\mathscr{A}_n$ and the minimality of F forces $F = F'$ and $F \in \mathscr{A}_n$.

Case II: $d(v) \leqslant n - 1$ for every vertex v in F non-adjacent to u_1. By Lemma 2, $K_{1,n} \subseteq F - u_1$, and this yields a vertex u_2 adjacent to u_1 with $d(u_2) = n + 1$. Let W be the set of common neighbors of u_1 and u_2 in F. Then $|W| \geqslant 1$ because of $2K_{1,n} \nsubseteq F$. Moreover, we may assume that all vertices w different from u_1 and u_2 with $d(w) \geqslant n$ belong to W since otherwise we would obtain a situation equivalent to Case I. At least two vertices $w_1, w_2 \in W$ with degree at least n must occur since otherwise there would be a cycle C_3 with u_1, u_2 and some $w \in W$ such that $K_{1,n} \nsubseteq F - E(C_3)$ in contradiction to Lemma 2. If w_1 and w_2 are adjacent then F contains a subgraph belonging to $\mathscr{B}_n$ which implies that $F \in \mathscr{B}_n$. It remains that w_1 and w_2 and all other vertices of degree $\geqslant n$ in W are non-adjacent. By Lemma 3 there must be a star in $F - u_1u_2$ containing the vertices u_1, u_2, w_1, w_2. This is only possible if a vertex w_3 adjacent to all four vertices exists. It follows that $4 \leqslant d(w_3) \leqslant n - 1$ yielding $n \geqslant 5$. But then F contains a subgraph belonging to $\mathscr{C}_n$ implying that $F \in \mathscr{C}_n$. $\square$

The following theorems will characterize the graphs in $\mathscr{R}''(2K_2, K_{1,n})$ with $n \geqslant 3$. Note that by Lemma 5 for $F \in \mathscr{R}''(2K_2, K_{1,n})$ with $F \neq 2K_{1,n}$ the number α of vertices of degree n satisfies $3 \leqslant \alpha \leqslant n + 2$.

Theorem 3. *Let $n \geqslant 3$ and let K_{n+2} be the complete graph with vertex-set $V = \{u_1, \ldots, u_{n+2}\}$. Let $E' = \{u_{2i-1}u_{2i}\colon 1 \leqslant i \leqslant \lfloor (n+2)/2 \rfloor\}$ and w be a vertex not belonging to V. Then*

$$H_n = \begin{cases} K_{n+2} - E' = K_{n+2} - \dfrac{n+2}{2} K_2 & \text{if } n \text{ is even,} \\ K_{n+2} - (E' \cup \{u_1 u_{n+2}\}) + wu_1 & \text{if } n \text{ is odd} \end{cases}$$

is the only graph in $\mathscr{R}''(2K_2, K_{1,n})$ with $\alpha = n+2$.

Proof. It is easy to see that $\Delta(H_n) = n$ and that H_n has $n+2$ vertices of degree n. Using Lemma 2 we obtain $H_n \to (2K_2, K_{1,n})$. For every edge e in H_n a vertex u exists which is adjacent to every vertex of degree n not incident to e. This implies $K_{1,n} \not\subseteq (H_n - e) - u$, and $H_n - e \nrightarrow (2K_2, K_{1,n})$ follows from Lemma 2(i). Thus, H_n belongs to $\mathscr{R}''(2K_2, K_{1,n})$, and it remains to show that $\mathscr{R}''(2K_2, K_{1,n})$ contains no further graph with $\alpha = n+2$.

Let $F = (V, E)$ be any graph in $\mathscr{R}''(2K_2, K_{1,n})$ with $\alpha = n+2$. Let $U = \{u_1, \ldots, u_{n+2}\}$ be the set of vertices of degree n in F and $W = V \setminus U$. One of the following two cases must occur:

Case I: $W = \emptyset$. Then $|V| = n+2$ and $d_{\overline{F}}(v) = 1$ for every $v \in V$. This implies that n is even and that F is isomorphic to H_n.

Case II: $W \neq \emptyset$. Let $w \in W$. By Lemma 1 there must be an edge from w to U and we may assume that $wu_1 \in E$. Then it follows from Lemma 3 that a star S containing $u_2, \ldots, u_{n+2}$ must exist. Note that the central vertex of any star containing $n+1$ vertices from U has to be one of the $n+1$ vertices. Thus, we may assume that u_{n+2} is the central vertex of the star S. Then $u_1 u_{n+2} \notin E$ because of $d(u_{n+2}) = n$. We distinguish the following two subcases:

(i) One of the vertices $u_2, \ldots, u_{n+1}$, say u_2, is adjacent to a vertex $w' \in W$ (it is allowed that $w = w'$). Using Lemma 3 we obtain that in $F - u_2 w'$ a star containing $u_1, u_3, u_4, \ldots, u_{n+2}$ must exist. Because of $u_1 u_{n+2} \notin E$ one of the vertices $u_3, \ldots, u_{n+1}$, say u_{n+1}, has to be the central vertex. It follows that $u_2 u_{n+1} \notin E$ because of $d(u_{n+1}) = n$. Again using Lemma 3 we obtain that $F - u_{n+1}u_{n+2}$ has to contain a star with $u_1, \ldots, u_n$ among its vertices. Let c be the central vertex of this star. By definition of W we obtain that $c \in U$. Because of $u_1 u_{n+2}, u_2 u_{n+1} \notin E$ it follows that $c \in \{u_1, \ldots, u_n\}$. But this yields $d(c) > n$, a contradiction.

(ii) The n neighbors of every u_i with $2 \leqslant i \leqslant n+2$ belong to U. This implies that every such u_i is not adjacent to exactly one other vertex in U. Note that $u_1 u_{n+2} \notin E$ and $u_i u_{n+2} \in E$ for $i = 2, \ldots, n+1$ has been already shown. Suppose first that u_1 is not adjacent to at least two vertices in $\{u_2, \ldots, u_{n+1}\}$, say to u_n and to u_{n+1}. It follows that $u_i u_{n+1} \in E$ for $i = 2, \ldots, n$. By Lemma 3, $F - u_{n+1}u_{n+2}$ has to contain a star with $u_1, \ldots, u_n$. Let c be the central vertex of this star. It follows that $c \in U$. Moreover, $c \in \{u_2, \ldots, u_{n-1}\}$ since u_1 is not adjacent to u_n, u_{n+1} and u_{n+2}. But then we have $n-1$ edges from c to $\{u_1, \ldots, u_n\}$ and two to u_{n+1} and u_{n+2}, a contradiction to $d(c) = n$. It remains that u_1 is not adjacent to at most one vertex in $\{u_2, \ldots, u_{n+1}\}$. Because of $u_1 w \in E$ and $d(u_1) = n$ there must be exactly $n-1$ neighbors of u_1 in $\{u_2, \ldots, u_{n+1}\}$

and $W=\{w\}$ because of Lemma 1. Let F' be the subgraph of F induced by U. Then u_1 has degree 2 in $\overline{F'}$ and u_i degree 1 for $i=2,\dots,n+2$. This implies that n is odd and $F=F'+u_1w$ has to be isomorphic to H_n. □

Theorem 4. *Let $n\geqslant 3$. Then for n even $\mathscr{R}''(2K_2,K_{1,n})$ does not contain a graph with $\alpha=n+1$. For n odd the only graphs in $\mathscr{R}''(2K_2,K_{1,n})$ with $\alpha=n+1$ are those obtained in the following way: Take a $K_{n+1}-[(n+1)/2]K_2$ with vertices $u_1,\dots,u_{n+1}$. Add k vertices $w_1,\dots,w_k$ with $2\leqslant k\leqslant n+1$, and $n+1$ edges $e_1,\dots,e_{n+1}$ such that (for $i=1,\dots,n+1$ and $j=1,\dots,k$) the edge e_i joins the vertex u_i and some vertex $w\in\{w_1,\dots,w_k\}$, and $1\leqslant d(w_j)\leqslant n-1$.*

Proof. It is easy to see that for n odd every graph obtained as decribed in the theorem belongs to $\mathscr{R}''(2K_2,K_{1,n})$ and has exactly $n+1$ vertices of degree n.

Let now $F=(V,E)$ be a graph in $\mathscr{R}''(2K_2,K_{1,n})$ with $\alpha=n+1$. Let $U=\{u_1,\dots,u_{n+1}\}$ be the set of vertices of degree n in F and let $W=V\backslash U$. Then Lemma 2 implies that every $u\in U$ is not adjacent to at least one vertex in $U\backslash\{u\}$. Thus, every $u\in U$ must have at least one neighbor in W.

First suppose that one vertex in U, say u_1, has at least two neighbors w_1 and w_2 in W. Note that the central vertex of any star with n vertices from U must be one of the n vertices and has to be different from u_1. Lemma 3 implies that $F-u_1w_1$ has to contain a star S_1 with $u_2,\dots,u_{n+1}$. We may assume that u_2 is the central vertex of S_1 and this yields $u_1u_2\notin E$. There is a neighbor w of u_2 in W, and Lemma 3 guarantees the existence a star S_2 in $F-u_2w$ with $u_1,u_3,\dots,u_{n+1}$. We may assume that u_3 is the central vertex of S_2. But this yields $d(u_3)\geqslant n+1$, a contradiction.

The remaining case is that every $u\in U$ has exactly one neighbor in W and is not adjacent to exactly one vertex in $U\backslash\{u\}$. This implies $n+1\equiv 0 \pmod 2$ and the subgraph induced by U is a $K_{n+1}-[(n+1)/2]K_2$. Moreover, $d(w)\leqslant n-1$ for every $w\in W$ by definition of W and this yields $|W|\geqslant 2$. Lemma 1 implies $d(w)\geqslant 1$ for every $w\in W$ and $E([W])=\emptyset$, and this yields $|W|\leqslant n+1$. Thus, F must have the structure of the graphs given in the theorem and the proof is complete. □

Theorem 5. *Let $n\geqslant 3$ and $F=(V,E)$ be a graph with exactly α vertices of degree n where $3\leqslant\alpha\leqslant n$. Let $U=\{v\in V\colon d(v)=n\}$ and $W=V\backslash U$. Moreover, let $U'=\{u\in U\colon K_{1,\alpha-2}\nsubseteq[U\backslash\{u\}]\}$ and $u_1,\dots,u_{\alpha'}$ be the vertices in U' if $\alpha'=|U'|\geqslant 1$. Then $F\in\mathscr{R}''(2K_2,K_{1,n})$ if and only if the following conditions hold*:

(i) $d_{[U]}(u)\leqslant\alpha-2$ *for every* $u\in U$,

(ii) $1\leqslant d(w)\leqslant\alpha-1$ *for every* $w\in W$,

(iii) $E([W])=\emptyset$,

(iv) *If $\alpha'\geqslant 1$ then there are α' vertices $w_1,\dots,w_{\alpha'}\in W$ such that $N(w_i)=U\backslash\{u_i\}$ for $i=1,\dots,\alpha'$, where $N(w_i)$ denotes the set of neighbors of w_i.*

Proof. First let $F\in\mathscr{R}''(2K_2,K_{1,n})$. Then Lemmas 1 and 2 imply (i)–(iii). To prove (iv) let $\alpha'\geqslant 1$ and $u_i\in U'$. Because of $d(u_i)=n$ an edge $u_iw\in E$ where $w\in W$

must exist. Then, Lemma 3 guarantees the existence of a star in $F-u_iw$ containing all vertices from $U_i=U\backslash\{u_i\}$. Since $K_{1,\alpha-2}\nsubseteq[U_i]$ by definition of U', the central vertex has to be a vertex $w_i\in W$. Thus, w_i is adjacent to all vertices in U_i and $N(w_i)=U_i$ follows using (ii) and (iii). Especially, $u_iw_i\notin E$ for $i=1,\dots,\alpha'$, and this implies $w_i\neq w_j$ for $1\leqslant i<j\leqslant\alpha'$.

Suppose now that conditions (i)–(iv) are satisfied. Then it follows from (i) and (ii) that a vertex $v\in V$ adjacent to all vertices of degree n in $V\backslash\{v\}$ does not exist. Thus, Lemma 2 implies $F\rightarrow(2K_2,K_{1,n})$. To prove the minimality of F note that every edge $e\in E$ has to be incident to a vertex $u\in U$ by (iii). If $e=uv$ with $u\in U'$ then a vertex $w\in W$ with $N(w)=U\backslash\{u\}$ exists by (iv). It follows that $K_{1,n}\nsubseteq(F-e)-w$, and Lemma 2 implies $F-e\nrightarrow(2K_2,K_{1,n})$. If $e=uv$ with $u\in U\backslash U'$ then $K_{1,\alpha-2}\subseteq[U\backslash\{u\}]$ by definition of U'. Let u^* be the central vertex of this star. Then $K_{1,n}\nsubseteq(F-e)-u^*$, and again we obtain that $F-e\nrightarrow(2K_2,K_{1,n})$. Thus, $F\in\mathscr{R}''(2K_2,K_{1,n})$ is proved and the proof of the theorem is complete. $\square$

Theorem 6. *The members of* $\mathscr{R}(2K_2,K_{1,3})$ *are the graphs* $F_1,\dots,F_{16}$ *in Fig.* 1.

Proof. Note that $\mathscr{R}(2K_2,K_{1,3})=\mathscr{R}'(2K_2,K_{1,3})\cup\mathscr{R}''(2K_2,K_{1,3})$. Using Theorem 2 we obtain $\mathscr{R}'=\{F_1,\dots,F_4\}$. Lemma 5 implies that $F_{16}=2K_{1,3}\in\mathscr{R}''$ and that $3\leqslant\alpha\leqslant5$ for the other members of $\mathscr{R}''$. Using Theorem 3 we then get F_5 as the only member with $\alpha=5$, whereas Theorems 4 and 5 yield $F_6,\dots,F_{10}$ as the members with $\alpha=4$ and $F_{11},\dots,F_{15}$ as the members with $\alpha=3$. $\square$

Of course, $\mathscr{R}(2K_2,K_{1,n})$ could be determined explicitly for some further small n by using Theorems 2–5, but applying Theorem 5 becomes much more tedious.

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